General Topology and its Applications 4 (1974) 51–72. © North-Holland Publishing Company

FREE TOPOLOGICAL GROUPS

Barbara V. Smith THOMAS

Department of Mathematics, Memphis State University, Memphis, Tenn. 38142, USA

Received 12 December 1972

Abstract: Main Theorem. *The free abelian topological group over a Tychonoff*[1] *space contains as a closed subspace a homeomorphic copy of each finite power of the space.*

A major and immediate corollary of this theorem is: *If $\mathcal{P}$ is a closed-hereditary property of Tychonoff spaces, and if the free abelian topological group over a Tychonoff spaces has $\mathcal{P}$, then so does every finite power of the space.* In particular, the corollary shows that the following properties are not preserved by passage to the free abelian group: normal, k-, sequential, Fréchet, Lindelöf, paracompact, pseudocompact, countably compact, sequentially compact, etc.

AMS Subj. Class.: Primary 22A05, 18A40; Secondary 20E05, 54D10, 54D15, 54D20, 54D50, 54D55

free topological group — free abelian topological group

0. Introduction

The Main Theorem arose in the following way. In 1968, Wilansky asked Rajagopalan if a separable, Hausdorff, normal topological group need be Lindelöf. Franklin and Rajagopalan constructed an example of a separable, Hausdorff, normal, but not Lindelöf topological space [1], and deduced that if the free abelian topological group over this space were normal, then they would be able to answer Wilansky's question in the negative. This leads naturally to the question of the preservation of normality by the free abelian group. The corollary to the Main Theorem of this paper shows that the free abelian topological group over a normal space need not be normal; this does not answer Wilansky's question, but does close the simplest avenue of approach.[2] Another example of a

[1] By "Tychonoff" we mean Hausdorff and completely regular.

[2] The free abelian topological group over the space constructed by Franklin and Rajagopalan may still be normal, but if so it will depend on further properties of the space, not just its normality.

separable, Hausdorff, normal, but not Lindelöf topological space was given by Rudin [15]. An example of a separable, Hausdorff, but not Lindelöf topological group appears as an exercise in [18, p. 251].

The notions of free topological groups and free abelian topological groups were introduced by Markov in 1941 [11]. He showed that every Tychonoff space has a free topological group which is Hausdorff, and which contains as a closed subspace a homeomorphic copy of the original space. Markov's proof involves a complicated construction of multinorms on the free group over the underlying set of the space.

Kakutani [8] and Samuel [16] have provided proofs of the existence of free topological groups, and free abelian topological groups, over any topological space. Their approach was to construct a continuous function from a space to a subgroup of a product of topological groups (analogous to one construction of the Stone–Čech compactification) with the usual factorization property implied by the word "free". They used what Freyd later called a "solution set" [2], for a discussion of this approach see [7, pp. 72–73]. Wyler has shown, using his notion of a top category, that all "free algebra" functors from the category of sets and functions lift to "free topological algebra" functors from the category of topological spaces and continuous functions [19]. Wyler's approach has the advantage that it informs us that the underlying algebraic structure of the free topological algebra over a space is just the free algebra over the underlying set of the space.

Using the technique of factoring appropriately chosen continuous functions through the free topological groups over a space, we obtain the following theorems.

Theorem 0.1. *The free topological group and free abelian topological group over a topological space are Hausdorff if and only if the space is functionally Hausdorff.* [3]

Theorem 0.2. *The universal morphisms from a topological space to its free topological group and free abelian topological group are closed embeddings if and only if the space is Tychonoff.*

These results are improvements on Markov's results, but that is not the reason for their inclusion here. They are included because the proofs

[3] Recall that a space is *functionally Hausdorff* if every pair of distinct points can be separated by a continuous real valued function.

obtained by lifting continuous functions are much simpler than Markov's original proofs mentioned above.

In Section 1 of this paper we present known results which will be used later, including the Basic Existence Theorem for free topological groups. In Section 2 we prove Theorem 0.1; Section 3 contains the proof of Theorem 0.2 and other elementary properties of free topological groups. We next show, in Section 4, the rather surprising result that *lifted real valued functions on the free abelian topological group need not determine its topology,* and we develop some additional machinery to be used in the proof of the Main Theorem, which comprises Section 5.

To avoid confusion, the notation $f^{\leftarrow}[\ ]$ will be used for the inverse image of a set under the function f, reserving the symbol f^{-1} for the inverse function or inverse in the group under consideration. The identity element in a group (abelian group) G will be denoted by e_G (0_G).

1. Preliminaries

Definition 1.1. The *free topological group* over a topological space X consists of a topological group $F(X)$ and a continuous function $\eta_X: X \to F(X)$, with the property that for any continuous function f from X to a topological group G there is a unique continuous group homomorphism $\hat{f}: F(X) \to G$, so that

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & F(X) \arrow[dl, "\hat{f}"] \\ & G & \end{tikzcd}$$

commutes.

By substituting "abelian topological group" for each occurrence of "topological group" in the above definition one obtains the definition of the *free abelian topological group* over a space X, denoted by $\eta_X : X \to Z(X)$.

The following two lemmas lead directly to the Basic Existence Theorem.

Lemma 1.2. *If* $\{\tau_\alpha\}_{\alpha\in A}$ *is a family of topologies on a group* F *compatible with the group structure* [4], *then the topology with subbase* $\bigcup_\alpha \tau_\alpha$ *is compatible with the group structure of* F.

[4] A topology is *compatible with the group structure* if the group operations are continuous.

Lemma 1.3. *If $\hat{f}\colon F \to G$ is a group homomorphism and τ is a topology on G compatible with the group structure, then the topology $\hat{f}^{\leftarrow}[\tau] = \{\hat{f}^{\leftarrow}[U] : U \in \tau\}$ is compatible with the group structure of F.*

Theorem 1.4 (Basic Existence Theorem). *For any topological space X, the free topological group $\eta_X : X \to F(X)$ and the free abelian topological group $\eta_X : X \to Z(X)$ exist.*

Proof sketch for $\eta_X : X \to F(X)$. Let the underlying group of $F(X)$ be the free group on the underlying set of X. Provide $F(X)$ with the finest topology compatible with the group structure, so that $\eta_X : X \to F(X)$, $\eta_X(x) = x^1$, is continuous (Lemma 1.2). Now any continuous function $f\colon X \to G$, where G is a topological group, surely lifts to a group homomorphism $\hat{f}\colon F(X) \to G$. Using Lemma 1.3 and the fact that $\hat{f} \circ \eta_X = f$, we conclude that $\hat{f}$ is, in fact, continuous.

The Basic Existence Theorem can be restated as: *The forgetful functors from the categories of topological groups and abelian topological groups to the category of topological spaces have left adjoints F and Z, respectively.*

It is well known that a left adjoint functor such as Z is unique up to isomorphism. This fact leads to the observation that $Z(X)$ is isomorphic (as a topological group) to $F(X)/C$, where C is the commutator subgroup of $F(X)$, and $F(X)/C$ is given the quotient topology of the canonical $q\colon F(X) \to F(X)/C$. The proof of this observation consists of noticing that for an abelian topological group A and a continuous function $f\colon X \to A$ the following diagram can be uniquely completed.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & F(X) \arrow[r, "q"] \arrow[d, "\hat{f}"] & F(X)/C \arrow[dl, dashed] \\
& A &
\end{tikzcd}$$

It will be useful to fix some notation. The elements of the group $F(X)$ are reduced words $x_1^{i(1)} x_2^{i(2)} \ldots x_n^{i(n)}$, where each $x_k \in X$ and $i(k) \in \mathbf{Z} \setminus \{0\}$; multiplication is juxtaposition and reduction using the usual laws of exponents, and the identity element is the empty word. We will represent the elements of $Z(X)$ as formal sums $\Sigma_{x \in X} k_x x$ with each $k_x \in \mathbf{Z}$ and $k_x = 0$ for all but finitely many x. It will sometimes be useful to suppress the terms with coefficient 0 and to write the elements of the free abelian group as $\Sigma_{i=1}^{n} k_i x_i$. The sum of two elements $\Sigma_{x \in X} k_x x$ and $\Sigma_{x \in X} j_x x$ is $\Sigma_{x \in X} (k_x + j_x) x$, and the identity element is $0 = \Sigma_{x \in X} 0x$.

In order to prove Theorem 0.1, we will need two observations about functionally Hausdorff spaces.

Lemma 1.5. *A topological space X is functionally Hausdorff if and only if for every pair of distinct points x_1 and x_2 in X, there is a Tychonoff space Y and a continuous function f: $X \to Y$, so that $f(x_1) \neq f(x_2)$.*

Proof. If X is functionally Hausdorff, let $Y = \mathbf{R}$ and f be the function obtained from the definition of functionally Hausdorff. For the converse, embed Y in an appropriate product of copies of $\mathbf{R}$ and project to a copy of $\mathbf{R}$ where the coordinates of $f(x_1)$ and $f(x_2)$ differ.

Lemma 1.6. *Given n distinct points $x_1, x_2, \ldots, x_n$ in a functionally Hausdorff space X, and n (not necessarily distinct) real numbers r_1, $r_2, \ldots, r_n$, there is a continuous function h: $X \to \mathbf{R}$ such that $h(x_i) = r_i$, $i = 1, 2, \ldots, n$.*

2. Proof of Theorem 0.1

We will prove this theorem in two parts.

Theorem 2.1. *A topological space X is functionally Hausdorff if and only if $Z(X)$ is Hausdorff.*

Proof. If $Z(X)$ is Hausdorff, then it is Tychonoff; furthermore, $\eta_X : X \to Z(X)$ separates points of X. Thus, by Lemma 1.5, X is functionally Hausdorff. For the converse, it suffices to show that $0 = \Sigma_{x \in X} 0x$ is closed since topological groups are homogeneous and have the property that T_1 implies Hausdorff. So consider any element $\Sigma_{x \in X} k_x x \neq 0$ in $Z(X)$; write $\Sigma_{x \in X} k_x x$ as $\Sigma_{i=1}^{n} k_i x_i$ with each $k_i \neq 0$. By Lemma 1.6, there is a continuous function h: $X \to \mathbb{R}$, so that $h(x_1) = 1$ and $h(x_i) = 0$ for $i \neq 1$. Lift h to the continuous group homomorphism $\hat{h}$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "h"'] & & Z(X) \arrow[dl, "\hat{h}"] \\ & \mathbf{R} & \end{tikzcd}$$

Then $\hat{h}(\Sigma_{i=1}^n k_i x_i) = k_1 \neq 0$. It follows that

$$\hat{h}^{\leftarrow}[(-\infty, -\tfrac{1}{2}|k_1|) \cup (\tfrac{1}{2}|k_1|, \infty)]$$

is a neighborhood of $\Sigma_{i=1}^n k_i x_i$ in $Z(X)$ which misses 0.

Unfortunately, the same result for $F(X)$ is more complicated since group homomorphisms from $F(X)$ to **R** will not separate words which differ only by a commutator. We can, however, immediately conclude that

Corollary 2.2. *A topological space X is functionally Hausdorff if and only if the commutator subgroup of $F(X)$ is closed.*

Proof. The commutator subgroup of $F(X)$ is the kernel of the canonical quotient map $q: F(X) \to Z(X)$.

It follows from this corollary that if X is functionally Hausdorff, then every point of $F(X)$ lying outside the commutator subgroup has a neighborhood which misses the identity. It remains to show that any element of the commutator subgroup can be separated from $e_{F(X)}$. However, it appears less complicated to ignore the preceding corollary and prove directly that any element of $F(X)$ can be separated from $e_{F(X)}$.

Note that if $\omega \in F(X)$, $\omega \neq e_{F(X)}$, then the reduced representation $x_1^{i(1)} x_2^{i(2)} \dots x_n^{i(n)}$ of ω has the property that for every $k < n$, $x_k \neq x_{k+1}$.

Theorem 2.3. *A topological space X is functionally Hausdorff if and only if $F(X)$ is Hausdorff.*

Proof. If $F(X)$ is Hausdorff, then, as in Theorem 2.1, X must be functionally Hausdorff. For the converse, suppose $\omega \neq e_{F(X)}$ is an element of $F(X)$, $\omega = x_1^{i(1)} x_2^{i(2)} \dots x_n^{i(n)}$. We must temporarily make some simplifications; we consider the free group $FG(S)$ on the set $S = \{x_1, x_2, \dots, x_n\}$ and invoke a theorem of Neuman [14] that $FG(S)$ is isomorphic to a subgroup of $FG(\{x, y\})$, the free group on two generators. Call the injective homomorphism $\phi: FG(S) \to FG(\{x, y\})$.

Now suppose we can find a Hausdorff, pathwise connected topological group M and a group homomorphism $\psi: FG(\{x, y\}) \to M$, so that $\psi\phi(\omega) \neq e_M$. In M we can find a path $p: [0, 1] \to M$ connecting all the distinct $\psi\phi(x_k)$. We then choose one element r_k in each $p^{\leftarrow}[\psi\phi(x_k)]$, so that $r_l = r_k$ if $x_l = x_k$. By Lemma 1.6, there is a continuous

$h: X \to [0, 1]$, so that $h(x_k) = r_k$ for all k. Hence we can find a continuous function $f = ph: X \to M$, such that $f(x_k) = \psi\phi(x_k)$ for all k. This function f lifts to a continuous group homomorphism $\hat{f}: F(X) \to M$ such that

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & F(X) \arrow[dl, "\hat{f}"] \\ & M & \end{tikzcd}$$

commutes. This lifted function has the property that $\hat{f}(\omega) = \psi\phi(\omega) \neq e_M$. Thus we can find a neighborhood U of $\hat{f}(\omega)$ which misses e_M. It follows that $\hat{f}^{\leftarrow}[U]$ is a neighborhood of ω which misses $e_{F(X)}$ and the theorem is proved.

The remainder of the proof is devoted to showing that the desired M and ψ exist. The underlying group of M is the multiplicative group of infinite dimensional upper triangular real matrices with 1's on the diagonal; we will write the elements of M as $I + Z$, where Z is strictly upper triangular.

$$Z = \begin{pmatrix} 0 & z_{12} & z_{13} & \cdots \\ & 0 & z_{23} & \cdots \\ & & 0 & \\ \bigcirc & & & \ddots \end{pmatrix}.$$

Then $(I+Z)^{-1}$ can be written as $I - Z + Z^2 - Z^3 + \ldots$; we need not worry about convergence of this expression since $(Z^n)_{ij} = 0$ whenever $j - i < n$. Provide M with the product topology of $\mathbf{R}^{\aleph_0}$; M is then clearly pathwise connected and Hausdorff.

To show that multiplication $m: M \times M \to M$ and taking inverses $^{-1}: M \to M$ are continuous, it suffices to show that $\pi_{ij} \circ m: M \times M \to \mathbf{R}$ and $\pi_{ij} \circ {}^{-1}: M \to \mathbf{R}$ are continuous for each $\pi_{ij}: M \to \mathbf{R}$ with $i < j$, where $\pi_{ij}(I+Z) = (I+Z)_{ij}$. Consider first $\pi_{ij} \circ m: M \times M \to \mathbf{R}$. Suppose $((I+Z)(I+W))_{ij} = r$, then formally $r = \Sigma_{k=1}^{\infty} (I \times Z)_{ik}(I+W)_{kj}$. However, $I + W$ is column finite, so, in fact, for some n depending only on i and j, $r = \Sigma_{k=1}^{n} (I+Z)_{ik}(I+W)_{kj}$. Hence $\pi_{ij} \circ m: M \times M \to \mathbf{R}$ can be expressed as a (finitary) derived operation $\mathbf{R}^n \to \mathbf{R}$ and thus is continuous. Similarly, for any $(I+Z) \in M$, and $i < j$,

$$((I+Z)^{-1})_{ij} = \sum_{k=1}^{n} (-1)^k (Z^k)_{ij} ,$$

where n depends only on i and j, and thus $\pi_{ij} \circ {}^{-1} : M \to \mathbf{R}$ can also be expressed as a finitary derived operation on $\mathbf{R}$. It follows that M is a topological group.

To obtain the group homomorphism ψ: $FG(\{x, y\}) \to M$, choose two countable sequences $(x_k)_{k\in\mathbf{N}}$ and $(y_k)_{k\in\mathbf{N}}$, so that the family of all x_k and y_k is algebraically independent. Define g: $\{x, y\} \to M$ by

$$g(x) = \begin{pmatrix} 1 & x_1 & & & 0 \\ & 1 & x_2 & & \\ & & 1 & x_3 & \\ 0 & & & \ddots & \ddots \end{pmatrix} = I + X$$

and

$$g(y) = \begin{pmatrix} 1 & y_1 & & & 0 \\ & 1 & y_2 & & \\ & & 1 & y_3 & \\ 0 & & & \ddots & \ddots \end{pmatrix} = I + Y .$$

Then g lifts to a group homomorphism ψ: $FG(\{x, y\}) \to M$.

We must show that $\psi\phi(\omega) \neq I$. We first compute $X^{\alpha(1)} Y^{\beta(1)} \ldots X^{\alpha(n)} Y^{\beta(n)}$ for positive exponents $\alpha(k)$ and $\beta(k)$. Let

$$X^{\alpha(1)} Y^{\beta(1)} \ldots X^{\alpha(n)} Y^{\beta(n)} = B$$

and let

$$\alpha(1) + \beta(1) + \alpha(2) + \ldots + \alpha(n) + \beta(n) = d .$$

Then $B_{ij} = 0$ unless $j = i + d$ and

$$B_{i,i+d} = x_i x_{i+1} \ldots x_{i+\alpha(1)-1}\, y_{i+\alpha(1)}\, y_{i+\alpha(1)+\beta(1)-1}\, x_{i+\alpha(1)+\beta(1)} \cdots y_{i+d-1} .$$

Since all the x_k and y_k are nonzero, it follows that $B \neq 0$. Similar calculations show that we can allow $\alpha(1) = 0$ and/or $\beta(n) = 0$ and still have $B \neq 0$.

Now consider any homogeneous polynomial U_d of finite degree $d > 0$ in X and Y. As before, $(U_d)_{ij} = 0$ unless $j = i + d$, and if $j = i + d$, then $(U_d)_{ij}$ is a linear combination of the numbers $B_{i,i+d}$ corresponding to the monomials in U_d. The assumption that the x_k and y_k are algebraically independent allows us to conclude that each $(U_d)_{i,i+d} \neq 0$ and thus $U_d \neq 0$.

Finally, any sum of the form $\Sigma_{d=1}^{\infty} r_d U_d$, where each U_d is a homogeneous polynomial of degree d in X and Y and some $r_d \neq 0$, is nonzero because the U_d contribute to distinct superdiagonals and thus no cancellation can occur.

Since $\phi(\omega) = x^{i(1)}y^{j(1)} \ldots x^{i(n)}y^{j(n)} \neq e_{FG(\{x,y\})}$, it now suffices to show that

$$\psi\phi(\omega) = (I+X)^{i(1)}(I+Y)^{j(1)} \ldots (I+X)^{i(n)}(I+Y)^{j(n)}$$

can be written as $I + \Sigma_{d=1}^{\infty} r_d U_d \neq I$. Here we must allow $i(1)$ or $j(n)$ to be 0, but all the other exponents are nonzero. For the following argument we will assume that all exponents are nonzero; the other three cases are similar. Since for every $m \in \mathbb{N}$, we have

$$(I+X)^m = I + mX + X^2(h(X)) ,$$

for some polynomial $h(X)$, we must have

$$(I+X)^{-m} = I - mX + X^2(h(X)) ,$$

for some series $h(X)$. Hence

$$\psi\phi(\omega) = (I + i(1)X + X^2(h_1(X)))$$
$$\times (I + j(1)Y + Y^2(g_1(Y))) \ldots (I + j(n)Y + Y^2(g_n(Y))) .$$

This product contains a unique monomial of degree $2n$ with each exponent equal to 1, namely

$$\left(\prod_{k=1}^{n} i(k)\right)\left(\prod_{k=1}^{n} j(k)\right) XYXY \ldots XY .$$

Thus $U_{2n} \neq 0$, $r_{2n} \neq 0$ and therefore $\psi\phi(\omega) \neq I$.

Example 2.4. A regular Hausdorff space which is not functionally Hausdorff and whose free abelian topological group and free topological group are, therefore, not Hausdorff is Hewitt's example [6] of a regular space whose every continuous function to the real line is constant.

3. Further elementary properties of free groups

Proposition 3.1. *A topological space X is completely regular (without Hausdorff) if and only if $\eta_X : X \to Z(X)$ is an embedding.*

Proof. If X is homeomorphic to $\eta_X(X)$, then X must be completely regular since $Z(X)$ is. For the converse, suppose X is completely regular and consider $Z(\eta_X(X))$; by definition, $Z(X)$ and $Z(\eta_X(X))$ have isomorphic underlying groups. To see that they are isomorphic topological groups, consider

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d] \arrow[dr, phantom, "\divideontimes"] & Z(X) \arrow[d, dashed, "g"] \\
\eta_X(X) \arrow[r] & Z(\eta_X(X))
\end{tikzcd}$$

where $g(\Sigma_{x\in X} k_x x) = \Sigma_{x\in X} k_x(\eta_X(x))$, and

$$\begin{tikzcd}
\eta_X(X) \arrow[rr] \arrow[dr, "i"'] & \arrow[d, phantom, "\divideontimes"] & Z(\eta_X(X)) \arrow[dl, dashed, "h = \hat{i}"] \\
& Z(X) &
\end{tikzcd}$$

where i is the inclusion map and $h(\Sigma_{x\in X} k_x(\eta_x(x))) = \Sigma_{x\in X} k_x x$. Both induced maps are continuous group homomorphisms and $gh = \mathrm{id}_{Z(\eta_X(X))}$, $hg = \mathrm{id}_{Z(X)}$.

Now suppose we are given any continuous $f: X \to \mathbf{R}$; f lifts to $Z(X)$ and we obtain

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=4, "\divideontimes"] & Z(X) \arrow[d, "\hat{f}"] \arrow[dr, phantom, shift right=4, "\divideontimes"] & \eta_X(X) \arrow[l, "i"'] \arrow[dl, dashed, "f|_{\eta_X(X)}"] \\
& \mathbf{R} &
\end{tikzcd}$$

with $j\,|_{\eta_X(X)}(\eta_X(x)) = f(x)$. Similarly, for any continuous $f: \eta_X(X) \to \mathbf{R}$, we have

$$\begin{tikzcd}
\eta_X(X) \arrow[r] \arrow[drr, "f"'] & Z(\eta_X(X)) \arrow[r, "h"] \arrow[dr, "\hat{f}"] & Z(X) \arrow[d, "\hat{f}g"] & X \arrow[l, "\eta_X"'] \arrow[dl, "\hat{f}g\eta_X"] \\
 & & \mathbf{R} &
\end{tikzcd}$$

with $\hat{f}g\eta_X(x) = f(\eta_X(x))$. Hence X and $\eta_X(X)$ have the same continuous functions to $\mathbf{R}$ and being completely regular must, therefore, have the same topology.

Proposition 3.2. *A topological space X is completely regular if and only if $\eta_X: X \to F(X)$ is an embedding.*

Proof. The diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X"'] & F(X) \arrow[d, "q"] \\
 & Z(X)
\end{tikzcd}$$

commutes. By Proposition 3.1, if X is completely regular $\eta_X: X \to Z(X)$ is an embedding. Hence so also is $\eta_X: X \to F(X)$. The converse follows as before from the observation that $F(X)$ is completely regular.

The corollaries to the preceding propositions are perhaps more interesting than the propositions themselves.

Corollary 3.3. *For a functionally Hausdorff space X the following are equivalent:*

(i) *X is Tychonoff,*
(ii) *$\eta_X: X \to Z(X)$ is an embedding,*
(iii) *$\eta_X: X \to F(X)$ is an embedding.*

Corollary 3.4. *For a functionally Hausdorff space X both spaces $\eta_X(X)$ are the Tychonoff reflection* [5] *of X.*

[5] That is, for any continuous function $f: X \to Y$ where Y is Tychonoff there is a unique factorization

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f"'] & \eta_X(X) \arrow[d, dashed, "\hat{f}"] \\
 & Y
\end{tikzcd}$$

Proof of Theorem 0.2. Suppose X is Tychonoff; we have already shown that $\eta_X: X \to Z(X)$ is an embedding. It remains to show that $\eta_X(X)$ is a closed subspace of $Z(X)$. Consider

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "i"'] \arrow[dr, phantom, "\text{※}"] & Z(X) \arrow[d, "Z(i)"] \\ \beta X \arrow[r, "\eta_{\beta X}"'] & Z(\beta X) \end{tikzcd}$$

and note that $Z(i)$ is injective. Now since $\eta_{\beta X}$ is continuous and $Z(\beta X)$ is Hausdorff, $\eta_{\beta X}(\beta X)$ is a closed subset of $Z(\beta X)$. Suppose $(y_\delta)_{\delta \in D}$ is a net in $\eta_X(X)$ which converges to some y in $Z(X)$. Then $(Z(i)(y_\delta))_{\delta \in D}$ converges to $Z(i)(y)$ in $Z(\beta X)$. However, for all $\delta \in D$, $Z(i)(y_\delta) \in \eta_{\beta X}(\beta X)$ and hence $Z(i)(y) \in \eta_{\beta X}(\beta X)$. It follows that $y \in \eta_X(X)$ and $\eta_X(X)$ is closed in $Z(X)$.

We could repeat this proof for $F(X)$, but it is simpler to note that

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "\eta_X"'] & & F(X) \arrow[dl, "q"] \\ & Z(X) & \end{tikzcd}$$

commutes, and thus $\eta_X: X \to F(X)$ must be a closed embedding since $\eta_X: X \to Z(X)$ is.

For the converse, suppose $\eta_X: X \to Z(X)$ is a closed embedding. Then, by Proposition 3.1, X is completely regular. If X were not Hausdorff, there would be a net $(x_\delta)_{\delta \in D}$ in X which converges to two distinct points x_1 and x_2. Then in $Z(X)$ the net $(1x_\delta)_{\delta \in D} = (2x_\delta - 1x_\delta)_{\delta \in D}$ would converge to $2x_1 - 1x_2 \notin \eta_X(X)$. Hence X must be Hausdorff as well as completely regular. For $F(X)$ a similar argument yields $(x_\delta^1)_{\delta \in D}$ converging to $x_1^2 x_2^{-1} \notin \eta_X(X)$ and thus X must again be Tychonoff.

Proposition 3.5. *If X' is a subspace of the topological space X, then $Z(X')$ is algebraically a subgroup of $Z(X)$, and the topology of $Z(X')$ is no coarser than its subspace topology.*

Proof. Let $i: X' \hookrightarrow X$ denote the inclusion mapping, then the diagram

$$\begin{tikzcd} X' \arrow[r, "\eta_{X'}"] \arrow[d, "i"'] & Z(X') \arrow[d, "Z(i)"] \\ X \arrow[r, "\eta_X"'] & Z(X) \end{tikzcd}$$

commutes. Algebraically $Z(i)$ is an inclusion mapping, but it need not be an embedding.

Example 3.6. To see that the topology of $Z(X')$ may be strictly finer than the subspace topology of $Z(i)(Z(X'))$, consider $X = \omega_0 + 1$ and $X' = \omega_0$, both with their usual topology. In $Z(X)$ the sequence $(1(n))_{n\in\omega_0}$ converges to $1(\omega_0)$, so $(1(n) - 1(n+1))_{n\in\omega_0}$ converges to $1(\omega_0) - 1(\omega_0) = 0$ in $Z(i)(Z(X'))$. However, $Z(X')$ is discrete, so $(1(n) - 1(n+1))_{n\in\omega_0}$ does not converge in $Z(X')$.

There are three further observations that can be made at this point.

Proposition 3.7. *If X' is a retract of X, then $Z(X')$ is a retract of $Z(X)$.*

Proof. Z is a functor and functors preserve retracts.

Proposition 3.8. *If X' is a closed subspace of a Tychonoff space X, then the subgroup of $Z(X)$ generated by X' is closed.*

Proof. Call the subgroup G, then $\Sigma_{x\in X} k_x x \in G$ if and only if $\{x: k_x \neq 0\}$ is contained in X'. Suppose $\Sigma_{x\in X} k_x x \notin G$, rewrite $\Sigma_{x\in X} k_x x$ as $\Sigma_{i=1}^n k_i x_i$ with each $k_i \neq 0$ and all the x_i distinct. Then there is an i_0 such that $x_{i_0} \notin X'$; without loss of generality assume that $i_0 = 1$. Since X is Tychonoff, there is a continuous function $f: X \to \mathbf{R}$ with $f(x_1) = 1$ and $f(X' \cup \{x_i: i \neq 1\}) = 0$. Lift this function, obtaining

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & Z(X) \arrow[dl, "\hat{f}"] \\ & \mathbf{R} & \end{tikzcd} .$$

Now $\hat{f}(G) = 0$ and $\hat{f}(\Sigma_{i=1}^n k_i x_i) = k_1 \neq 0$, so $\hat{f}^{\leftarrow}[(k_1 - \frac{1}{2}, k_1 + \frac{1}{2})]$ is a neighborhood of $\Sigma_{i=1}^n k_i x_i$ which misses G.

Note that a closed subset of a topological group need not generate a closed subgroup; for example, $\{1/n : n \in \mathbf{N}\} \cup \{0\}$ is a closed subset of $\mathbf{R}$ and generates $\mathbf{Q}$.

Proposition 3.9. *Z preserves quotient maps, that is, if $q: X \to X'$ is a quotient map, then $Z(q): Z(X) \to Z(X')$ is a quotient map.*

Proof. In the category of topological spaces and in the category of abelian topological groups the quotient maps are precisely the onto maps $q: X \to X'$ satisfying the following property: For each commutative diagram

$$\begin{tikzcd}
A \arrow[r, "m"] & B \\
X \arrow[u, "f"] \arrow[r, "q"'] & X' \arrow[u, "g"']
\end{tikzcd}$$

with m a monomorphism, there is a unique $h: X' \to Y$ making everything commute (such a morphism q is called a *strong epimorphism* [9]).

Let $q: X \to X'$ be a quotient map and suppose that

$$\begin{tikzcd}
A \arrow[r, "m"] & B \\
Z(X) \arrow[u, "f"] \arrow[r, "Z(q)"'] & Z(X') \arrow[u, "g"']
\end{tikzcd}$$

is a commutative diagram in the category of abelian topological groups with m injective. Then we have

$$\begin{tikzcd}
A \arrow[rrr, "m"] & & & B \\
& X \arrow[ul, "f\eta_X"'] \arrow[r, "q"] \arrow[dl, "\eta_X"] & X' \arrow[ull, dashed, "h"'] \arrow[ur, "g\eta_{X'}"'] \arrow[dr, "\eta_{X'}"] & \\
Z(X) \arrow[uu, "f"] \arrow[rrr, "Z(q)"'] & & & Z(X') \arrow[uu, "g"']
\end{tikzcd}$$

with a continuous $h: X' \to A$ as indicated. Lift h to $\hat{h}: Z(X') \to A$, with $\hat{h}\eta_{X'} = h$. We show that $\hat{h}$ is the desired continuous group homomorphism "making everything commute". We use the fact that if two group homomorphisms agree on $\eta_X(X)$, or $\eta_{X'}(X')$, then they are equal. Chasing diagrams yields $\hat{h} \circ Z(q) \circ \eta_X = f\eta_X$ and $m \circ \hat{h} \circ \eta_{X'} = g\eta_{X'}$; thus $\hat{h} \circ Z(q) = f$ and $m \circ \hat{h} = g$.

4. Lifted real valued functions do not suffice

The major method used to prove properties of $Z(X)$ has been, and will continue to be, the lifting of real valued continuous functions on X. Since these functions serve to determine the topology of a completely regular space, it is rather surprizing that they do not determine the topology of the free abelian group over a completely regular space.

Theorem 4.1. *The free abelian topological group over a Tychonoff space need not have the weak topology of its continuous real valued group homomorphisms.*

Proof. We show that $Z(\mathbf{N})$ does not have the weak topology of its continuous real valued group homomorphisms. $Z(\mathbf{N})$ is discrete, so it suffices to show that $0 \in Z(\mathbf{N})$ is not the intersection of any finite number of inverse images of open sets of $\mathbf{R}$ under group homomorphisms, or, equivalently, for any additive function $f: Z(\mathbf{N}) \to \mathbf{R}^n$, $n \in \mathbf{N}$, and any neighborhood V of 0 in $\mathbf{R}^n$, $f^{\leftarrow}[V]$ contains some point in addition to 0 in $Z(\mathbf{N})$. If we represent elements of $Z(\mathbf{N})$ as infinite columns with integer entries

$$\sum_{j \in \mathbf{N}} k_j j = (k_1, k_2, \ldots k_j, \ldots)^{\mathrm{T}} ,$$

then any additive $f: Z(\mathbf{N}) \to \mathbf{R}^n$ can be represented by a matrix A_f with n infinite rows.

$$A_f = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \\ a_{n1} & a_{n2} & \cdots \end{pmatrix} .$$

Let $\mathbf{R}^{n+1}$ be represented by infinite columns of real numbers with all entries x_j equal to 0 if $j > n+1$.

$$x = (x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots)^{\mathrm{T}} .$$

Then $A_f \cdot y = 0$ for some $y \neq 0$ in $\mathbf{R}^{n+1}$. Now, given any neighborhood V of 0 in $\mathbf{R}^n$, there is a symmetric, convex neighborhood U of 0 in $\mathbf{R}^{n+1}$ such that $A_f \cdot U \subseteq V$. Furthermore, $\bigcup_{\lambda \in \mathbf{R}} (\lambda y + U)$ is a symmetric, convex neighborhood of 0 in $\mathbf{R}^{n+1}$ with infinite volume and with the property that

$$A_f \cdot \Big(\bigcup_{\lambda \in \mathbf{R}} (\lambda y + U)\Big) \subseteq V .$$

By Minkowski's Convex Body Theorem [12, Section 30], there is a $k \in \bigcup_{\lambda \in R} (\lambda y + U)$, $k \neq 0$, with integral coefficients. Considering this k as an element of $Z(\mathrm{N})$, we find that $k \in f^{\leftarrow}[V]$. Thus $f^{\leftarrow}[V] \setminus \{0\} \neq \emptyset$, and $Z(\mathrm{N})$ does not have the weak topology of its continuous group homomorphisms.

The previous theorem indicates that it is necessary to develop some additional machinery to examine the topological structure of $Z(X)$. The next two lemmas are analogues of Theorem 1.4 and Proposition 3.1, and have similar proofs. The first is also a special case of Wyler's results [19]. They lead to the crucial Proposition 4.4.

Lemma 4.2. *The forgetful functor from the category of topological real vector spaces to the category of topological spaces has a left adjoint V.*

Lemma 4.3. *If X is a Tychonoff space, then $\eta_1 : X \to V(X)$ is an embedding.*

Proposition 4.4. *For any Tychonoff space X and any nonzero integer z, $zX = \{zx : x \in X\} \subseteq Z(X)$ is homeomorphic to X.*

Proof. It suffices to show this proposition for positive integers since $(-n)X = -(nX)$ which is homeomorphic to nX.

Note that all finitary derived operations in any category of topological algebras are continuous, and in particular, $n(-)$: $Z(X) \to Z(X)$ is continuous. By the construction of $V(X)$, $1/n\,(-)$: $V(X) \to V(X)$ is continuous.

Consider the following commutative diagram of continuous functions:

$$\begin{tikzcd}
 & Z(X) \arrow[r, dashed, "n(-)"] \arrow[d, "\hat{\eta}_1"] & Z(X) \arrow[d, "\hat{\eta}_1"] \\
X \arrow[ur, "\eta_X"] \arrow[r, "\eta_1"'] & V(X) & V(X) \arrow[l, "\frac{1}{n}(-)"]
\end{tikzcd}$$

Here $nX = n(\eta_X(X))$ and $1/n \circ \hat{\eta}_1(nX) = \eta_1(X) \cong X$ and thus nX is homeomorphic to X.

5. Proof of the Main Theorem

Proposition 5.1. *For any topological space X, $Z(X)$ is the disjoint union of the homeomorphic clopen subspaces $Z_p(X) = \{\Sigma_{x \in X} k_x x \colon \Sigma_{x \in X} k_x = p\}$, $p \in \mathbf{Z}$.*

Proof. Lift the function $f\colon X \to \mathbf{Z}$, $f(x) = 1$ for all $x \in X$, to $\hat{f}\colon Z(X) \to \mathbf{Z}$.

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=4ex, "\divideontimes"] & & Z(X) \arrow[dl, "\hat{f}"] \\
 & \mathbf{Z} &
\end{tikzcd}$$

Then $\hat{f}(\Sigma_{x \in X} k_x x) = \Sigma_{x \in X} k_x$ and thus $Z_p(X) = \hat{f}^{\leftarrow}\{p\}$ is clopen. It is clear that the sets $Z_p(X)$ are disjoint; they are homeomorphic since each is a coset of $Z_0(X)$.

Proposition 5.2. *For a Tychonoff space X the set $Y_m = \{\Sigma_{x \in X} k_x x \colon$ at most m coefficients k_x are nonzero$\}$ is a closed subset of $Z(X)$.*

Proof. For an element $z = \Sigma_{i=1}^{n} k_i x_i$ with more than m nonzero coefficients, there are pairwise disjoint neighborhoods $U_1, U_2, \ldots, U_{m+1}$ of $x_1, x_2, \ldots, x_{m+1}$, respectively, and continuous functions $f_j\colon X \to \mathbf{R}$, $j = 1, 2, \ldots, m+1$, such that $f_j(x_j) = 1$ and $f_j((X \setminus U_j) \cup \{x_i\}_{i=m+2}^{n}) = 0$. All these functions lift to $\hat{f}_j\colon Z(X) \to \mathbf{R}$, with $\hat{f}_j(z) = k_j$, $j = 1, 2, \ldots, m+1$.

The open set

$$V = \bigcap_{j=1}^{m+1} \hat{f}_j^{\leftarrow}[(k_j - \tfrac{1}{2}, k_j + \tfrac{1}{2})]$$

is a neighborhood of z. For any element $y \in V$ and for any $j = 1, 2, \ldots, m+1$, $\hat{f}_j(y) \neq 0$, which implies that there is an $x_j' \in U_j$ such that x_j' has nonzero coefficient in y. Since the U_j are disjoint, it follows that y has at least $m+1$ nonzero coefficients. Hence $V \cap Y_m = \emptyset$.

Corollary 5.3. *For a Tychonoff space X, the set of elements of $Z(X)$ with one nonzero coefficient is closed.*

Proof. The desired set is $Y_1 \setminus \{0\} = Y_1 \setminus Z_0(X)$.

Proof of the Main Theorem. We first show by induction that X^n can be embedded as a subspace of $Z(X)$. By Proposition 3.1, $X_1 = \eta_X(X) = \{1x: x \in X\}$ is homeomorphic to X. For each $n \in \mathbf{N}$, let

$$X_n = \left\{ \sum_{i=1}^{n} 2^{i-1} x_i : (x_1, \ldots, x_n) \in X^n \right\},$$

and assume that X_{n-1} is homeomorphic to X^{n-1}. We will show that X_n is homeomorphic to $X_{n-1} \times X \cong X^n$. Define $\pi_n : X_n \to X$ by $\pi_n(\Sigma_{i=1}^n 2^{i-1} x_i) = x_n$. We must first assure ourselves that π_n is well defined. As long as all the x_i are distinct there is no problem. If the x_i are not distinct, then we can write $\Sigma_{i=1}^n 2^{i-1} x_i$ as $\Sigma_{j=1}^m a_j x_j'$ with all the x_j' distinct, and x_n is the unique x_j' with $a_j \geqslant 2^{n-1}$.

Now, algebraically each $y \in X_n$ has a unique presentation $y = y' + 2^{n-1}\eta_X(\pi_n(y))$ with $y' \in X_{n-1}$. We must show that the bijective mapping $(y', x) \mapsto y' + 2^{n-1}\eta_X(x)$ is a homeomorphism of $X_{n-1} \times X$ onto X_n. For continuity forward, observe that this mapping is the composition

$$\begin{array}{ccccc} X_{n-1} \times X & \xrightarrow{\mathrm{id} \times \eta_X} & X_{n-1} \times \eta_X(X) & \xrightarrow{\mathrm{id} \times 2^{n-1}(-)} & X_{n-1} \times 2^{n-1} X \\ & & & & \big\downarrow + \\ & & & & X_n \end{array}$$

For continuity in the other direction, it suffices to show that π_n is continuous, for then $y \mapsto y' = y - 2^{n-1}\eta_X(\pi_n(y))$ is continuous and thus $y \mapsto (y', \pi_n(y))$ is continuous.

To show that π_n is continuous let F be a closed subset of X and consider

$$\pi_n^{\leftarrow}[F] = \left\{ \sum_{i=1}^{n} 2^{i-1} x_i : x_n \in F \right\} .$$

If $\Sigma_{i=1}^n 2^{i-1} y_i \notin \pi_n^{\leftarrow}[F]$, then $y_n \notin F$, so there is a continuous function $f: X \to \mathbf{R}$ such that $f(y_n) = 1$, $f(F) = 0$, and $0 \leqslant f(x) \leqslant 1$ for all $x \in X$. This function lifts to $\hat{f}: Z(X) \to \mathbf{R}$. Now

$$\hat{f}\left(\sum_{i=1}^{n} 2^{i-1} y_i\right) = \sum_{i=1}^{n} 2^{i-1} f(y_i) \geqslant 2^{n-1} ,$$

and for all $\Sigma_{i=1}^n 2^{i-1} x_i \in \pi_n^{\leftarrow}[F]$,

$$\begin{aligned} \hat{f}\left(\sum_{i=1}^{n} 2^{i-1} x_i\right) &= \sum_{i=1}^{n} 2^{i-1} f(x_i) = \sum_{i=1}^{n-1} 2^{i-1} f(x_i) \\ &\leqslant \sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1} - 1. \end{aligned}$$

It follows that $\hat{f}^{\leftarrow}[(2^{n-1} - \frac{1}{2}, \infty)] \cap X_n$ is an X_n neighborhood of $\Sigma_{i=1}^n 2^{i-1} y_i$ which misses $\pi_n^{\leftarrow}[F]$.

To show that X_n is a closed subspace of $Z(X)$ we also use an induction argument. By Theorem 0.2, $X_1 = \eta_X(X)$ is closed in $Z(X)$. Assume that X_{n-1} is a closed subset of $Z(X)$ and consider some point $\Sigma_{y \in X} k_y y \in \mathrm{cl}_{Z(X)} X_n$. Then $\Sigma_{y \in X} k_y y$ has at most n nonzero coefficients (Proposition 5.2) and $\Sigma_{y \in X} k_y = 2^n - 1$ (Proposition 5.1). Hence there is at least one nonzero k_y and we can write $\Sigma_{y \in X} k_y y$ as $\Sigma_{j=1}^m k_j y_j$ with $1 \leqslant m \leqslant n$, all the y_j distinct, and all the k_j nonzero.

Furthermore, there must be a net $(\Sigma_{i=1}^n 2^{i-1} x_{i,\delta})_{\delta \in D}$ in X_n which converges in $Z(X)$ to $\Sigma_{j=1}^m k_j y_j$. Now $(x_{n,\delta})_{\delta \in D}$ must cluster at some y_j. If not, for each $j = 1, 2, ..., m$, there is a $\delta(j) \in D$ such that $\mathrm{cl}_X(\{x_{n,\delta}\}_{\varepsilon \geqslant \delta(j)})$ does not contain y_j. Let δ_0 be greater than $\delta(j)$ for $j = 1, 2, ..., m$; it follows that $\mathrm{cl}_X(\{x_{n,\delta}\}_{\delta \geqslant \delta_0}) \cap \{y_j\}_{j=1}^m = \emptyset$. Hence there is a continuous function $f: X \to \mathbf{R}$ such that $f(y_j) = 0$ for $j = 1, 2, ..., m$, $f(\mathrm{cl}_X\{x_{n,\delta}\}_{\delta \geqslant \delta_0}) = 1$, and $0 \leqslant f(x) \leqslant 1$ for all $x \in X$. Lift f to $\hat{f}: Z(X) \to \mathbf{R}$; then for all $\delta \geqslant \delta_0$,

$$\hat{f}\left(\sum_{i=1}^{n} 2^{i-1} x_{i,\delta}\right) = \sum_{i=1}^{n} 2^{i-1} f(x_{i,\delta}) \geq 2^{n-1},$$

and

$$\hat{f}\left(\sum_{j=1}^{m} k_j y_j\right) = 0,$$

which contradicts the fact that $(\Sigma_{i=1}^{n} 2^{i-1} x_{i,\delta})_{\delta \in D}$ converges to $\Sigma_{j=1}^{m} k_j y_j$.

Since $(x_{n,\delta})_{\delta \in D}$ clusters at some y_j, say y_{j_0}, there is a sunset $(x_{n,\beta})_{\beta \in B}$ which converges (in X) to y_{j_0}. It follows that in $Z(X)$, $(2^{n-1} x_{n,\beta})_{\beta \in B}$ converges to $2^{n-1} y_{j_0}$ and $(\Sigma_{i=1}^{n} 2^{i-1} x_{i,\beta})_{\beta \in B}$ converges to $\Sigma_{j=1}^{m} k_j y_j$. Hence, since + is continuous,

$$\left(\sum_{i=1}^{n-1} 2^{i-1} x_{i,\beta}\right)_{\beta \in B} = \left(\left(\sum_{i=1}^{n} 2^{i-1} x_{i,\beta}\right) - 2^{n-1} x_{n,\beta}\right)_{\beta \in B}$$

converges to $\Sigma_{j=1}^{m} k_j y_j - 2^{n-1} y_{j_0}$. However, for all $\beta \in B$, $\Sigma_{i=1}^{n-1} 2^{i-1} x_{i,\beta} \in X_{n-1}$, which we have assumed is closed. Thus

$$\left(\sum_{j=1}^{m} k_j y_j\right) - 2^{n-1} y_{j_0} \in X_{n-1}.$$

Therefore, $\Sigma_{j=1}^{m} k_j y_j \in X_n$, and X_n is closed.

Corollary 5.4. *If* P *is a closed hereditary property of Tychonoff spaces and if X is a Tychonoff space, then X^n has property* P *whenever $Z(X)$ has property* P.

Example 5.5. The free abelian topological group over the Sorgenfrey line is not normal. The free abelian topological group over Q with the integer points identified is not a k-space.

Added in proof. Professor Sidney Morris has pointed out the following.

(1) Proposition 3.8 is a special case of Theorem 2.5 of S. Morris, Varieties of topological groups II, Bull. Austral. Math. Soc. 2 (1970) 1–13.

(2) Theorem 4.1 is a corollary of the main theorem of S. Morris, Locally compact abelian groups and the variety of topological groups generated by the reals, Proc. Am. Math. Soc. 34 (1972) 290–292.

(3) Another proof of the fact that finitely additive but not finitely productive properties of topological spaces are not preserved by the functor Z goes as follows: Let $X_1, X_2, ..., X_n$ be spaces such that $\amalg_n X_n$ has the property but $\Pi_n X_n$ does not. Then $Z(\amalg_n X_n) \cong \Pi_n Z(X_n)$, and $\Pi_n X_n$ is a closed subspace of $\Pi_n Z(X_n)$.

Acknowledgments

The author wishes to thank Professor O. Wyler for guiding her doctoral dissertation, from which the results of this paper were taken, and Professor S.P. Franklin for suggesting the problem of whether or not the free abelian topological group over a normal space need be normal. I also wish to thank the University of Pittsburgh for granting me an Andrew Mellon Postdoctoral Fellowship for the academic year 1972–1973.

References

[1] S.P. Franklin and M. Rajagopalan, Some examples in topology, Trans. Am. Math. Soc. 155 (1971) 305–314.

[2] P. Freyd, Abelian categories (Harper and Row, New York, 1964).

[3] B.R. Gelbaum, Free topological groups, Proc. Am. Math. Soc. 12 (1961) 737–743.

[4] M.I. Graev, Free topological groups, Izv. Akad. Nauk SSSR Ser. Mat. 12 (1948) 279–324 (in Russian; English Transl.: Am. Math. Soc. Transl. 35 (1951). Reprint Am. Math. Soc. Transl. (1) 8 (1962) 305–364).

[5] H. Herrlich, Topologische Reflexionen und Coreflexionen, Lecture Notes in Math. 78 (Springer, Berlin, 1968).

[6] E. Hewitt, On two problems of Urysohn, Ann. of Math. 47 (1946) 503–509.

[7] E. Hewitt and K. Ross, Abstract harmonic analysis I (Academic Press, New York, 1963).

[8] S. Kakutani, Free topological groups and infinite discrete product topological groups, Proc. Imp. Acad. Tokyo 20 (1944) 595–598.

[9] G.M. Kelley, Monomorphisms, epimorphisms, and pullbacks, J. Austral. Math. Soc. 9 (1969) 124–142.

[10] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory: Presentations of groups in terms of generators and relations, Series in Pure and Applied Mathematics, Vol. XIII (Interscience, New York, 1966).

[11] A.A. Markov, On free topological groups, C.R. (Doklady) Acad. Sci. URSS (N.S.) 31 (1941) 299–301; Bull. Acad. Sci. URSS Ser. Mat. (Izv. Akad. Nauk. SSSR) 9 (1945) 3–64 (in Russian, English Summary), English Transl.: Am. Math. Soc. Transl. 30 (1950) 11–88, Reprint Am. Math. Soc. Transl. (1) 8 (1962) 195–272.

[12] H. Minkowski, Geometrie der Zahlen (Leipzig and Berlin, 1910).

[13] S.A. Morris, Varieties of topological groups and left adjoint functors, J. Austral. Math. Soc. (1973) 220–227.

[14] B.H. Neumann, An essay on free products of groups with amalgamations, Philos. Trans. Roy. Soc. London 246 (1954) No. 919, 503–554.
[15] M.E. Rudin, A separable normal nonparacompact space, Proc. Am. Math. Soc. 7 (1956) 940–941.
[16] P. Samuel, On universal mappings and free topological groups, Bull. Am. Math. Soc. 54 (1948) 591–598.
[17] F.R. Ward, On free and projective topological groups, Notices Am. Math. Soc. 17 (1970) 135.
[18] A. Wilanski, Topology for analysis (Ginn, Waltham, Mass., 1970).
[19] O. Wyler, On the categories of general topology and topological algebra, Arch. Math. XXII (1971) 7–17
[20] O. Wyler, Top categories and categorical topology, Gen. Topology Appl. 1 (1971) 17–28.